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Ferromagnetic ordering in graphs with arbitrary degree distribution

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Abstract. We present a detailed study of the phase diagram of the Ising model in random graphs with arbitrary degree distribution. By using the replica method we compute exactly the value of the critical temperature and the associated critical exponents as a function of the moments of the degree distribution. Two regimes of the degree distribution are of particular interest. In the case of a divergent second moment, the system is ferromagnetic at all temperatures. In the case of a finite second moment and a divergent fourth moment, there is a ferromagnetic transition characterized by non-trivial critical exponents. Finally, if the fourth moment is finite we recover the mean field exponents. These results are analyzed in detail for power-law distributed random graphs.

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1 Introduction

The increasing evidence that many physical, biological and social networks exhibit topological properties different from that of random graphs or regular lattices has led to the investigation of graph models with complex topological properties [1]. In particular, the existence of some special nodes of the cluster (hubs) posses a larger probability to develop connections pointing to other nodes has been recently identified in scale-free networks [2,3]. These networks exhibit a power law degree distribution $p_k \sim k^{-\gamma}$, where the exponent γ is usually larger than 2. This kind of degree distribution implies that each node has a statistically significant probability of having a large number of connections compared to the average degree of the network. Examples of such properties can be found in communication and social webs, along with many biological networks, and have led to the developing of several dynamical models aimed to the description and characterization of scale-free networks [2–5].

Power law degree distributions are the signature of degree fluctuations that may alter the phase diagram of physical processes as in the case of random percolation [6,7] and spreading processes [8] that do not exhibit a phase transition if the degree exponent is $\gamma \leq 3$. In this perspective, it is interesting to study the ordering dynamics of the Ising model in scale-free networks. The Ising model is, indeed, the prototypical model for the study of phase transitions and complex systems and it is often the starting point for the developing of models aimed at the characterization of ordering phenomena. For this reason, the Ising model and its variations are used to mimic a wide range of problems not pertaining to physics, such as the forming and spreading of opinions in societies and companies or the evolution and competition of species. Since social and biological networks are often characterized by scale-free properties, the study of the ferromagnetic phase transition in graphs with arbitrary degree distribution can find useful application in the study of several complex interacting systems and it has been recently pursued in reference [9]. The numerical simulations reported in reference [9] show that in the case of a degree distribution with $\gamma = 3$ the Ising model has a critical temperature T_c, characterizing the transition to an ordered phase, which scales logarithmically with the network size. Therefore, in the thermodynamic limit, the system is ferromagnetic at any temperature.

In the present paper we present a detailed analytical study of the Ising model in graphs with arbitrary degree distribution. By relaxing the degree homogeneity in the usual mean field (MF) approach to the Ising model, it is shown that the existence of a disordered phase is related to the ratio of the first two moments of the degree distribution. Motivated by this finding, we apply the replica calculation method, as developed for spin glasses and diluted ferromagnetic models on random graphs [12–18], in order to find an exact characterization of the transition to the ordered state and its associated critical behavior.

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We find that a disordered phase is allowed only if the second moment of the degree distribution is finite. In the opposite case, the strong degree of the hubs present in the network prevails on the thermal fluctuations, imposing a long-range magnetic order for any finite value of the temperature. Corrections to this picture are found when the minimal allowed degree is $m = 1$. The value of the critical temperature and exponents is found for any degree exponent $\gamma > 3$ and a transition to the usual infinite dimensional MF behavior is recovered at $\gamma = 5$. Moreover, in the range $3 < \gamma \leq 5$ non trivial scaling exponents are obtained.

During the completion of the present work we become aware that Dorogovtsev *et al.* [10] have obtained with a different approach results which overlap with those reported in the present paper.

2 Mean field approach

Let us consider the Ising model with a ferromagnetic coupling constant on top of a random graph of size N and an arbitrary degree distribution p_k. A spin variable $s_i = \pm 1$ is assigned to each node i while the interactions among different nodes is given by the interaction matrix elements J_{ij} ($J_{ij} = 1$ if nodes i and j interact and $J_{ij} = 0$ otherwise). The energy of this system is given by

$$\mathcal{H}(\{s_i\}) = M - \sum_{i>j=1}^{N} J_{ij} s_i s_j - H_0 \sum_i s_i, \qquad (1)$$

where $M = \langle k \rangle N/2$ and H_0 is an external field.

Using a MF approximation it is possible to obtain a first estimate of the critical temperature and the magnetization taking into account the inhomogeneity of the graph. Neglecting the spin-spin correlations, $\langle s_i s_j \rangle \simeq \langle s_i \rangle \langle s_j \rangle = \langle s \rangle^2$, it is then possible to use an effective field ansatz in which each spin feels the average magnetization on neighboring spins obtaining

$$\mathcal{H}_{MF} = M - \langle s \rangle \sum_i k_i s_i, \qquad (2)$$

where $k_i = \sum_j J_{ij}$ is the node degree and the external filed has been set to $H_0 = 0$. In the case of a graph with homogeneous degree $k_i = \langle k \rangle$, the average magnetization is found self-consistently obtaining $\langle s \rangle = \tanh(\beta \langle k \rangle \langle s \rangle)$ where β is the inverse temperature in units of k_B^{-1}.

In the case of complex heterogeneous networks, we can relax the homogeneity assumption on the node's degree by defining the average magnetization $\langle s \rangle_k$ for the class of nodes with degree k. Indeed, the node's magnetization is strongly affected by the local degree and the homogeneity assumption results to be too drastic especially in singular degree distributions. The self-consistent equation for the average magnetization in each degree class reads simply as $\langle s \rangle_k = \tanh(\beta k \langle u \rangle)$, where $\langle u \rangle$ is now the effective field magnetization seen by each node on the nearest neighbors. In the calculation of the effective field we have to take into account the system's heterogeneity by noticing that each link points more likely to nodes with higher degree. In particular the probability that a link points to a node with degree k is $kp_k / \sum_l lp_l$. Thus, the correct average magnetization seen on a nearest neighbor node is given by

$$\langle u \rangle = f(\langle u \rangle) = \sum_k \frac{kp_k}{\langle k \rangle} \tanh(\beta k \langle u \rangle). \qquad (3)$$

Once obtained $\langle u \rangle$ it is possible to compute the network average magnetization as the average over all the degree classes $\langle s \rangle = \sum_k p_k \tanh(\beta k \langle u \rangle)$. A non-zero magnetization solution is obtained whenever $f'(0) > 1$ and when $f'(0) = 1$ we obtain the critical point that defines

$$T_c = \beta_c^{-1} = \frac{\langle k^2 \rangle}{\langle k \rangle}. \qquad (4)$$

Hence, when $\langle k^2 \rangle / \langle k \rangle$ is finite there is a finite critical temperature signalling of the transition from the paramagnetic to a ferro-magnetic state. On the contrary, if $\langle k^2 \rangle$ is not finite the system is always in the ferromagnetic state.

3 The replica approach on general random graphs

In the present section we will refine the mean field picture *via* a replica calculation, in the framework of the method applied in the last years for spin glasses and diluted ferromagnetic models on random graphs [13]. We will show how this method allows to calculate the value and condition for the existence of the critical temperature. Moreover, these results recover the MF predictions in the limits where the latter is applicable.

For a random graph the interaction matrix elements in equation (1) follow the distribution $P(J_{ij}) = (1 - \frac{\langle k \rangle}{N})\delta(J_{ij}) + \frac{\langle k \rangle}{N}\delta(J_{ij} - 1)$ with constraints in order to impose the correct degree distribution that will be introduced along the computation of the logarithm of the partition function. Following the approach of reference [15], we compute the free energy of the model with the replica method, exploiting the identity $\log \langle Z^n \rangle = 1 + n \langle \log Z \rangle + O(n^2)$. The average over the disorder of Z^n is given by

$$\langle Z^n \rangle = \sum_{s_i} \left\langle \exp\left[-\beta \sum_{a=1}^{n} \mathcal{H}(\{s_i^a\})\right] \right\rangle \qquad (5)$$

where $\langle \ldots \rangle$ is the average over the quenched interaction elements:

$$\langle A \rangle = \frac{1}{\mathcal{N}} \int \prod_{i<j} \mathrm{d}J_{ij} P(J_{ij}) \prod_{i=1}^{N} \delta\left(\sum_j J_{ij} - k_i\right) A \qquad (6)$$

$$\mathcal{N} = \prod_{i<j} \mathrm{d}J_{ij} P(J_{ij}) \prod_{i=1}^{N} \delta\left(\sum_j J_{ij} - k_i\right). \qquad (7)$$

Notice that the delta functions enforce the constraints on the connectivities distribution and $\mathcal{N}$ is a normalization factor. To compute the average over $\{J_{ij}\}$ we write the constraints in the integral form

$$\delta\left(\sum_j J_{ij}-k_i\right)=\int\frac{\mathrm{d}\psi_i}{2\pi}\mathrm{e}^{\mathrm{i}\left(\sum_j J_{ij}-k_i\right)\psi_i} \tag{8}$$

resulting that

$$\langle Z^n\rangle=\frac{\mathrm{e}^{-\beta nM-M}}{\mathcal{N}}\sum_{\boldsymbol{s}_i}\int\prod_i\left(\frac{\mathrm{d}\psi_i}{2\pi}\mathrm{e}^{-\mathrm{i}k_i\psi_i}\right)$$
$$\times\exp\left(\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{\beta\sum_a s_i^a s_j^a+\mathrm{i}(\psi_i+\psi_j)}+\beta H_0\sum_{i,a}s_i^a\right)\cdot \tag{9}$$

We now introduce a functional order parameter, following the well tested procedure of replica theory of diluted systems [13]:

$$\rho(\boldsymbol{\sigma})=\frac{1}{N}\sum_i\delta(\boldsymbol{\sigma}-\boldsymbol{s}_i)\mathrm{e}^{\mathrm{i}\psi_i}\cdot \tag{10}$$

Tracing over the spins $\boldsymbol{s}_i$, integrating out the ψ_i variables and computing the normalization factor $\mathcal{N}$ one is left with the following expression for the replica free energy:

$$\begin{aligned}-n\beta F &= -\langle k\rangle\sum_{\boldsymbol{\sigma}}\rho(\boldsymbol{\sigma})\hat\rho(\boldsymbol{\sigma})+\frac{\langle k\rangle}{2}(1-n\beta)\\ &+\frac{\langle k\rangle}{2}\sum_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}\rho(\boldsymbol{\sigma}_1)\rho(\boldsymbol{\sigma}_2)\exp\left(\beta\sum_a\sigma_1^a\sigma_2^a\right)\\ &+\sum_k p_k\log\left[\sum_{\boldsymbol{\sigma}}(\hat\rho(\boldsymbol{\sigma}))^k\mathrm{e}^{\beta H_0\sum_a\sigma^a}\right]\end{aligned} \tag{11}$$

where $\hat\rho(\boldsymbol{\sigma})$ is the functional order parameter conjugate to $\rho(\boldsymbol{\sigma})$. The main contribution to the free energy in the thermodynamic limit is evaluated *via* the following functional saddle point equations:

$$\rho(\boldsymbol{\sigma})=\sum_k\frac{kp_k}{\langle k\rangle}\frac{(\hat\rho(\boldsymbol{\sigma}))^{k-1}\mathrm{e}^{\beta H_0\sum_a\sigma^a}}{\sum_{\boldsymbol{\sigma}}(\hat\rho(\boldsymbol{\sigma}))^k\mathrm{e}^{\beta H_0\sum_a\sigma^a}} \tag{12}$$

$$\hat\rho(\boldsymbol{\sigma})=\sum_{\boldsymbol{\sigma}_1}\rho(\boldsymbol{\sigma}_1)\exp\left(\beta\sum_a\sigma^a\sigma_1^a\right). \tag{13}$$

It is easy to show that the order parameters can be taken as normalized in the $n\to 0$ limit. Further normalization factors cancel out in the expression for the free energy.

3.1 Solution of the saddle point equations

Being the system a diluted ferromagnet the replica symmetric (RS) ansatz is sufficient to find the correct solution of (12) and (13) at every temperature [13]. In the general case we can write:

$$\rho(\boldsymbol{\sigma})=\int\mathrm{d}hP(h)\frac{\mathrm{e}^{\beta h\sum_{a=1}^n\sigma_a}}{(2\cosh(\beta h))^n} \tag{14}$$

$$\hat\rho(\boldsymbol{\sigma})=\int\mathrm{d}uQ(u)\frac{\mathrm{e}^{\beta u\sum_{a=1}^n\sigma_a}}{(2\cosh(\beta u))^n} \tag{15}$$

that leads to

$$P(h)=\sum_k\frac{kp_k}{\langle k\rangle}\int\prod_{t=1}^{k-1}\mathrm{d}u_tQ(u_t)\delta\left(h-\sum_t u_t-H_0\right) \tag{16}$$

$$Q(u)=\int\mathrm{d}hP(h)\delta\left[u-\frac{1}{\beta}\tanh^{-1}(\tanh(\beta)\tanh(\beta h))\right] \tag{17}$$

where $P(h)$ is the average probability distribution of effective fields acting on the sites and $Q(u)$ is that of cavity fields. We would like to stress that the strong inhomogeneities present in the graph are correctly taken into account and handled *via* the computation of the whole probability distributions. In the Ising case we can easily work only with the cavity fields, whose self consistent equation for the $Q(u)$ reads:

$$\begin{aligned}Q(u) &= \sum_k\frac{kp_k}{\langle k\rangle}\int\prod_{t=1}^{k-1}\mathrm{d}u_tQ(u_t)\delta\Bigg\{u-\frac{1}{\beta}\tanh^{-1}\\ &\times\left[\tanh(\beta)\tanh\left(\beta\sum_t^{k-1}u_t+\beta H_0\right)\right]\Bigg\}\cdot\end{aligned} \tag{18}$$

This is an integral equation that can be solved at every value of β using a population dynamics algorithm such as the RS simple version of that proposed in [14]. Moreover, plugging equations (14) and (15) into (11) we obtain the following expression for the free energy

$$\begin{aligned}\beta F &= \langle k\rangle\int\mathrm{d}h\mathrm{d}uP(h)Q(u)\log\left[1+\tanh(\beta h)\tanh(\beta u)\right]\\ &-\frac{\langle k\rangle}{2}\int\prod_{t=1}^{2}\mathrm{d}h_tP(h_t)\log\left[1+\tanh(\beta)\prod_{t=1}^{2}\tanh(\beta h_t)\right]\\ &-\sum_k p_k\int\prod_{t=1}^{k}\mathrm{d}u_tQ(u_t)\log\left(\frac{2\cosh(\beta\sum_t u_t+\beta H_0)}{\prod_t 2\cosh(\beta u_t)}\right)\\ &-\langle k\rangle\log(2)+\frac{\langle k\rangle}{2}\left[\beta-\log(\cosh(\beta))\right]\cdot\end{aligned} \tag{19}$$

Then using equations (18, 19) we can compute the different thermodynamic quantities. For instance, the average cavity field and magnetization are given by

$$\langle u\rangle = \int \mathrm{d}u Q(u) u = \sum_k \frac{k p_k}{\langle k\rangle} \int \prod_{t=1}^{k-1} \mathrm{d}u_t Q(u_t) \times \frac{1}{\beta}\tanh^{-1}\left[\tanh(\beta)\tanh\left(\beta\sum_t^{k-1} u_t + \beta H_0\right)\right] \quad (20)$$

$$\langle s\rangle = -\frac{\partial F}{\partial H_0} = \sum_k p_k \int \prod_{t=1}^{k} \mathrm{d}u_t Q(u_t) \times \tanh\left(\beta\sum_t^{k} u_t + \beta H_0\right). \quad (21)$$

On the other hand, the internal energy and the specific heat at the saddle point can be calculated from equation (19) through the relations $\langle E\rangle = \beta\frac{\partial\beta F}{\partial\beta}$ and $C = \mathrm{d}\langle E\rangle/\mathrm{d}T$ and further exploiting (12) and (13).

3.2 Ferromagnetic phase transition

At $T = 0$ and in the limit of non vanishing fields (u and $h \sim O(1)$) it is straightforward to see that the cavity fields can take only 0 or 1 values, *i.e.* $Q(u) = q_0\delta(u) + (1 - q_0)\delta(u-1)$. Plugging this ansatz into equations (18, 20) and (21) one obtains that $\langle u\rangle = 1 - q_0$,

$$\langle s\rangle = 1 - G_0(q_0), \quad (22)$$
$$q_0 = G_1(q_0), \quad (23)$$

where

$$G_0(x) = \sum_k p_k\, x^k,$$
$$G_1(x) = \sum_k \frac{k p_k}{\langle k\rangle} x^{k-1}, \quad (24)$$

are the generating functions of the degree distributions of a vertex chosen at random and a vertex arrived following an edge chosen at random [21], respectively. We point out that these equations correctly coincides with that obtained in the problem of percolation in a random graph with an arbitrary degree distribution [20,21], where the average magnetization $\langle s\rangle$ is just the size of the giant component. Moreover, these expressions can be easily generalized to higher order hypergraphs as it has been done in [15–17]. From equation (22) it follows that there is a finite magnetization whenever the solution q_0 of equation (23) is less than 1. This happens whenever

$$\frac{\langle k^2\rangle}{\langle k\rangle} \geq 2, \quad (25)$$

that is just the condition for percolation in a random graph [20,21]. On the contrary, for $\langle k^2\rangle/\langle k\rangle < 2$ the magnetization (the size of giant component) is 0, *i.e.* the system is in a paramagnetic state.

For random graphs satisfying the percolation condition in equation (25) we are now interested in finding the value of β_c for the ferromagnetic transition. In the general case we can derive both sides of equation (20) in $u = 0$ self consistently, obtaining

$$\frac{1}{T_c} = \beta_c = -\frac{1}{2}\log\left(1 - 2\frac{\langle k\rangle}{\langle k^2\rangle}\right). \quad (26)$$

In the limit $\langle k^2\rangle \gg 2\langle k\rangle$ we can expand the logarithm getting the first order condition $T_c = \langle k^2\rangle/\langle k\rangle$ which is the value found in the naive mean field approximation (4). Hence, the MF approach in developed in the previous section is valid for $\langle k^2\rangle \gg 2\langle k\rangle$ and, in this case, it gives the same results as those obtained using the replica approach.

3.3 Critical behavior around β_c

The critical behavior of the thermodynamical quantities $\langle s\rangle$, χ, and δC close to β_c and of $\langle s\rangle$ at $\beta = \beta_c$ can be calculated without having to explicitly solve the self consistent equations for the whole probability distribution $Q(u)$. Sufficiently close to the critical point we can assume $Q(u) \sim \delta(u - \langle u\rangle)$ being $\langle u\rangle$ infinitesimal. In fact this ansatz is incorrect if $\beta > \beta_c$, because it correctly takes into account the connectivity distribution but disregards the non trivial structure of the $Q(u)$, which does not merely translate from the critical form $\delta(u)$ at β_c, but immediately develops a continuum structure. In the zero temperature limit the continuum shape will again collapse in a distribution of delta peaks discussed above. Nevertheless, sufficiently close to the transition we can expect only the first momenta of the $Q(u)$ to be relevant. For distributions with $\langle k^4\rangle$ finite one is left with a closed system of equations for the first three momenta all contributing to the same leading order. Defining $\mu_n = \langle k(k-1)...(k-n)\rangle$ and $A = ((\tanh(\beta))^2\mu_2)/(\beta^2\langle k\rangle - (\tanh(\beta))^2\mu_1)$ it follows that

$$\langle u\rangle = \frac{\tanh(\beta)}{\tanh(\beta_c)}\langle u\rangle - \frac{\beta^2\tanh(\beta)[1-(\tanh(\beta)]^2}{3\langle k\rangle} \times\left[\mu_1\langle u^3\rangle + 3\mu_2\langle u\rangle\langle u^2\rangle + \mu_3\langle u\rangle^3\right]$$
$$\langle u^2\rangle = A\langle u\rangle^2$$
$$\langle u^3\rangle = \left(\frac{(\tanh(\beta))^3 A\mu_2 + \mu_3}{\beta^3\langle k\rangle - (\tanh(\beta))^3 < \mu_1}\right)\langle u\rangle^3. \quad (27)$$

Similar calculations can be done for the free energy, the energy and the specific heat. The same equations are also found for $\langle k^4\rangle = \infty$, where the calculation is a bit more involved because the leading momenta are to be found *via* an analytic continuation in the values of their order. The corrections to the leading momenta are important to compute the the amplitudes in the scaling relations, because more terms at the same leading order are present, as we see in equation (27). Since we are not interested in the calculation of the amplitudes we can therefore resort to the variational ansatz $Q(u) \sim \delta(u - \langle u \rangle)$ in the proximity of the transition. However we would like to stress that calculations can be done also in the general case. Equations (20, 21) for the means then become

$$\langle u \rangle \sim \sum_k \frac{kp_k}{\langle k \rangle} \frac{1}{\beta} \tanh^{-1}[\tanh(\beta)\tanh(\beta(k-1)\langle u \rangle + \beta H_0)] \tag{28}$$

$$\langle s \rangle \sim \sum_k p_k \tanh(\beta k \langle u \rangle + \beta H_0). \tag{29}$$

The corresponding expressions for the free energy, the energy and the specific heat can be retrieved in the same way and will not be written here for the sake of space. If $\langle k^4 \rangle$ is finite the first non trivial term of the power series expansion of equation (28) that still gives an analytic contribution is simply $\langle u \rangle^3$. One finds

$$\langle u \rangle \sim \left(\frac{3 \langle k \rangle}{\beta_c^2 (\tanh \beta_c) \langle k(k-1)^3 \rangle} \right)^{\frac{1}{2}} \tau^{\frac{1}{2}} \tag{30}$$

$$\langle s \rangle \sim \langle u \rangle, \quad \chi \sim \tau^{-1}, \quad \langle s \rangle \sim H_0^{1/3} \tag{31}$$

where $\tau = 1 - T/T_c$ is the reduced control parameter. All exponents are the usual mean field ones. However, one finds a finite jump in the specific heat. The transition is therefore first order in the traditional sense. If we keep all the relevant momenta in our calculation, we find the expected correction to the amplitudes. For example we find $\langle u \rangle \sim \sqrt{3}((\beta_c \tanh(\beta_c) \langle k \rangle)((\mu_1 + 3\mu_2)A + \mu_3 >))^{-\frac{1}{2}} \tau^{\frac{1}{2}}$. This equation reduces to (30) if we disregard higher momenta.

4 Power law distributed graphs

In the following we are mostly interested in the case of a power law distribution of the type

$$p_k = c\, k^{-\gamma}, \qquad m \leq k < \infty, \tag{32}$$

where c is a normalization constant and m is the lowest degree. Note that in the case of a power law distribution

$$\langle k^2 \rangle = c \sum_{k=m}^{k_{max}} k^{2-\gamma} > c\, m \sum_{k=m}^{k_{max}} k^{1-\gamma} = m \langle k \rangle \cdot \tag{33}$$

Hence, we have that for $m \geq 2$ the graph is always percolating for all γ independently on the cutoff k_{max}. Then for $m \geq 2$ the critical temperature is always given by equation (26). However, for $m = 1$ there is a critical value $\gamma^\star$ beyond which the graph is no longer percolating [19]. $\gamma^\star$ is the solution of the equation $\langle k^2 \rangle = 2 \langle k \rangle$ resulting $\gamma^\star = 3.47875...$. If $\gamma \geq \gamma^\star$ the system is always paramagnetic while for $\gamma < \gamma^\star$ there is a transition to a ferromagnetic state at a temperature given by equation (26). In Figure 1 we show the phase diagram together with the

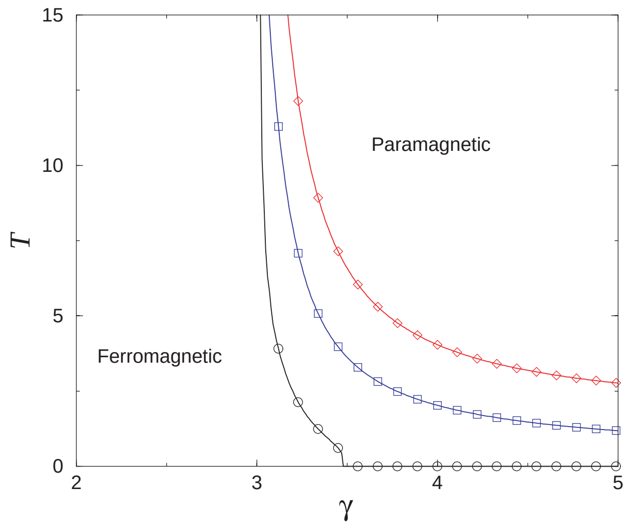

Fig. 1. The phase diagram of the Ising model on scale-free graphs with a power law degree distribution $p_k = ck^{-\gamma}$, $m \leq k < \infty$. The ferromagnetic transition lines depends on the value of m, with $m = 1$ circles, 2 squares, and 3 diamonds.

critical lines for $m = 1$, 2 and 3. For $\gamma > 5$ the fourth moment $\langle k^4 \rangle$ is finite and therefore we recover the usual MF exponents obtained in the previous section. In the next section we investigate the critical behavior for $\gamma < 5$.

4.1 Degree exponent $2 < \gamma \leq 3$

For $2 < \gamma \leq 3$ the second moment of the degree distribution diverges and, therefore, as discussed in previous sections, the system is always in a ferromagnetic state. In this case it is important to investigate the behavior of $\langle u \rangle$ and $\langle s \rangle$ when $\beta \to 0$. This computation can be done using either the mean-field or the replica approach obtaining the same results. In fact, in this case we have $\lim_{\beta \to 0} Q(u) = \delta(u)$ and putting this limit distribution into the self consistent equation for $\langle u \rangle$ and $\langle s \rangle$ we recover the mean field asymptotic behavior. For $2 < \gamma \leq 3$ the sums in equation (3) are dominated by the large k region and, therefore, they can be approximated by integrals resulting

$$\langle u \rangle \approx (\gamma - 2)(m\beta \langle u \rangle)^{\gamma-2} \int_{m\beta\langle u \rangle}^{\infty} \mathrm{d}x x^{1-\gamma} \tanh x, \tag{34}$$

while the magnetization, $\langle s \rangle = \sum_k p_k \tanh(\beta k \langle u \rangle)$, is simply given by

$$\langle s \rangle \approx \frac{\gamma - 1}{\gamma - 2} m\beta \langle u \rangle \cdot \tag{35}$$

For $\gamma = 3$ the integral in the rhs of equation (34) is dominated by the small x behavior. Thus, approximating the $\tanh x$ by x and computing $\langle u \rangle$ we obtain

$$\langle u \rangle \approx \frac{\exp(-1/m\beta)}{m\beta}, \qquad \gamma = 3. \tag{36}$$

Table 1. Asymptotic behavior of different thermodynamic quantities in the limit $T \to \infty$ for scale free networks with $2 < \gamma \leq 3$.

	$2 < \gamma < 3$	$\gamma = 3$
δC	$(1/T)^{(\gamma-1)/(3-\gamma)}$	$T^2 \mathrm{e}^{-2T/m}$
$\langle s \rangle$	$(m/T)^{1/(3-\gamma)}$	$\mathrm{e}^{-T/m}$
χ	$1/Tm^2$	T/m^2

On the other hand, for $\gamma < 3$ the integral in the rhs of equation (34) is finite for any value of $m\beta \langle u \rangle$ and, therefore, for $m\beta \langle u \rangle \ll 1$ it follows that

$$\langle u \rangle \approx [(\gamma-2)I]^{\frac{1}{3-\gamma}} (m\beta)^{\frac{\gamma-2}{3-\gamma}}, \quad \gamma < 3, \tag{37}$$

where $I = \int_0^\infty \mathrm{d}x x^{1-\gamma} \tanh x$. Finally, substituting equations (36, 37) in equation (35) we obtain the asymptotic behavior of $\langle s \rangle$. With the same technique one can study the behavior of the other thermodynamic quantities obtaining the asymptotic behaviors shown in Table 1.

The limiting case $\gamma = 3$ corresponds with the Barabasi-Albert model studied in [9] by means of numerical simulations. The magnetization exhibits an exponential decay in agreement with our calculation (see Tab. 1). Moreover, the critical temperature was observed to increase logarithmically with the network size N. Computing T_c in equation (4) for $\gamma = 3$ we obtain $T_c \approx (m/2) \ln N$, which is in very good agreement with their numerical results. It is worth remarking that similar exponential and logarithmic dependencies have been observed for the order and control parameter in some non-equilibrium transitions [8,11].

4.2 Degree exponent $3 < \gamma \leq 5$

In this case $\langle k^2 \rangle$ is finite and, therefore, there is a ferromagnetic transition temperature given by equation (26). However, $\langle k^4 \rangle$ is not finite and the derivation of the MF critical exponents performed in Section 3.3 is not valid. In order to find the critical exponents we can write the functions inside the connectivities sums as power series in $\langle u \rangle$. The coefficients of the two series will depend on the higher momenta of the connectivity distribution and will be infinite beyond a certain power of $\langle u \rangle$. This is direct consequence of the fact that the power expansion of the $\tanh(y)$ around 0 is convergent as long as the $y < \pi/2$, while for any $\langle u \rangle$ in our cases there will exist an $k^\star$ such that $y = \beta(k^\star - 1)\langle u \rangle + \beta H_0$ lays outside the convergence radius. Nevertheless, the function is well approximated by the expansion when one truncates it up to the maximum analytical value of the exponent such that all momenta of the power law distribution taken into consideration are finite.

For $3 < \gamma < 5$ the highest analytical exponent of the expansion of equation (28) in powers of $\langle u \rangle$ is $n_{max} = \gamma - 2$, where the integer value has been analytically continued and so should be done with the corresponding series coefficient. In this range of γ, n_{max} is lower than 3 and, therefore, it should be taken as the correct value instead of $n = 3$ that leads to non analicities. With analogous calculations we are able to find all other critical exponents (see Tab. 2). On the other hand, for $\gamma = 5$ one can find a logarithmic correction to the previous values expanding the inverse hyperbolic tangent in equation (28) to the third order in the tails of the degree distribution. The results are shown in Table 2. The specific heat is continuous at the critical point indicating a phase transition of order larger than 1. Moreover, a part from the logarithmic corrections in the $\gamma = 5$ case, the universality relations between the exponents are satisfied. We point out that non-trivial exponents have been also obtained in the problem of percolation in a power law random graph with $3 < \gamma \leq 5$ [22].

Table 2. Scaling of different thermodynamic quantities near the critical point $\tau \ll 1$. $\star$ indicates the presence of logarithmic corrections given in the text.

		$3 < \gamma < 5$	$\gamma = 5$	$\gamma > 5$
$\delta C \sim \tau^\alpha$	α	$\frac{5-\gamma}{\gamma-3}$	$\frac{1}{-\log \tau}$	1st order
$\langle s \rangle_\tau \sim \tau^\beta$	β	$\frac{1}{\gamma-3}$	$\frac{1}{2}^\star$	1/2
$\chi \sim \tau^{-\gamma}$	γ	1	1	1
$\langle s \rangle_{H_0} \sim H_0^{1/\delta}$	δ	$\gamma - 2$	$3^\star$	3

It is also interesting to analyze how these results are affected if there is a cutoff in the degree distribution, due to finite size effects for instance. In this case $\langle k^4 \rangle$ is finite and, therefore, we should recover the MF exponents obtained in the previous section. However the influence of non trivial terms is very strong and thus equation (30) is only valid in a very narrow region around T_c. The numerical values of T_c and of the amplitudes in the critical behavior of the magnetization are also strongly affected because they depend on moments of the connectivities distribution. In the infinite cutoff limit the mean field window shrinks to zero and one recovers the non trivial behavior. Indeed, if we work with a large enough cutoff and compute the average magnetization for $\beta(k-1)\langle u \rangle (\beta) \sim \pi/2$ then we see a contribution in the magnetization that goes as $(\beta - \beta_c)^{1/(\gamma-3)}$. This region becomes dominant for large values of the cutoff.

5 Summary and conclusions

In summary, we have obtained the phase diagram of the Ising model on a random graph with an arbitrary degree distribution. Three different regimes are observed depending on the moments $\langle k^2 \rangle$ and $\langle k^4 \rangle$ of the distribution. For $\langle k^4 \rangle$ finite the critical exponents of the ferromagnetic phase transition coincides with those obtained from the simple MF theory. On the contrary, for $\langle k^4 \rangle$ not finite but $\langle k^2 \rangle$ finite we found non-trivial exponents that depend on the power law exponent of the degree distribution γ. Finally, for $\langle k^2 \rangle$ not finite the system is always in a ferromagnetic state.

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